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BOUNDS FOR INTERPOLATIONAL PATH OF POSITIVE OPERATORS

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ABSTRACT. In this report, we shall investigate estimates of the upper bounds for the ratio between interpolational paths by terms of a generalized Specht ratio. Among others, if A and B are positive invertible operators on a Hilbert space H satisfying $MI \geq A, B \geq mI > 0$ for some scalars m and M and put $h = \frac{M}{m}$, then for $r < s$ and $t \in [0,1]$

$$A\ m_{s,t}\ B \leq S(h,r,s) A\ m_{r,t}\ B$$

where $S(h,r,s)$ is a generalized Specht ratio.

1. INTRODUCTION

The theory of operator means for positive operators on a Hilbert space is established by Kubo and Ando [7] in 1980: A binary operation $A\ \sigma\ B$ in the cone of positive invertible operators is called an operator mean if the following conditions are satisfied:

monotonicity: $A \leq C$ and $B \leq D$ imply $A\ \sigma\ B \leq C\ \sigma\ D$,
upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\ \sigma B_n \downarrow A\ \sigma\ B$,
transformer inequality: $T^*(A\ \sigma\ B)T \leq (T^*AT)\ \sigma\ (T^*BT)$ for every operator T,
normalized condition: $A\ \sigma\ A = A$.

A key for the theory is that there is a one-to-one correspondence between the operator means σ and the nonnegative operator monotone functions $f(x)$ on $(0, \infty)$ with $f(1) = 1$ by the formula

$$f(x) = 1\ \sigma\ x \qquad \text{for all } x > 0,$$

or

$$A\ \sigma\ B = A^{\frac{1}{2}}(1\ \sigma\ A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \quad \text{for all } A, B \geq \varepsilon > 0.$$

We say that f is the representing function for σ. In this case, notice that $f(t)$ is operator monotone if and only if it is operator concave. An operator mean σ is said to be symmetric if $A\ \sigma\ B = B\ \sigma\ A$ for all positive invertible operators A and B.

Simple examples of operator means are the weighted arithmetic mean ∇_t and the weighted harmonic mean $!_t$ $(0 < t < 1)$ defined by

$$A\ \nabla_t\ B = (1-t)A + tB \quad \text{and} \quad A\ !_t\ B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$$

respectively. Another one is the geometric mean $\sharp$ which is just corresponding to the operator monotonicity of the square root. As a matter of fact, the t-power mean (the weighted geometric mean) $\sharp_t$, for $0 \le t \le 1$, are determined by the operator monotone function x^t;

$$A \sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

and the geometric mean $\sharp$ is defined as $A \sharp B = A \sharp_{\frac{1}{2}} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$.

E.Kamei and one of the authors discussed an operator version of the interpolational paths of Uhlmann's type and showed that the interpolational path σ_t generated by an operator mean σ is convex and differentiable and the derivative $d\sigma_t/dt|_{t=0}$ is a solidarity, see [6].

Following [4], for a symmetric operator mean σ, a parametrized operator mean σ_t is called an interpolational path for σ if it satisfies

(1) $A \sigma_0 B = A, \quad A \sigma_{1/2} B = A \sigma B \quad$ and $\quad A \sigma_1 B = B$

(2) $(A \sigma_p B)\sigma(A \sigma_q B) = A \sigma_{\frac{p+q}{2}} B$ for all $p, q \in [0,1]$

(3) the map $t \in [0,1] \mapsto A \sigma_t B$ is norm continuous for each A and B.

Typical interpolational means are so-called power means;

$$A\, m_r\, B = A^{\frac{1}{2}} \left(\frac{1 + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r}{2} \right)^{\frac{1}{r}} A^{\frac{1}{2}} \quad \text{for } r \in [-1,1]$$

and their interpolational paths are

$$A\, m_{r,t}\, B = A^{\frac{1}{2}} \left(1 - t + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}} \quad \text{for } t \in [0,1].$$

For each $r \in [-1,1]$, $A\, m_{r,t}\, B$ ($t \in [0,1]$) is a path from A to B via $A\, m_r\, B$. In particular,

$$\begin{aligned} A\, m_{1,t}\, B &= A \nabla_t B = (1-t)A + tB, \\ A\, m_{0,t}\, B &= A \sharp_t B, \\ A\, m_{-1,t}\, B &= A\, !_t\, B = ((1-t)A^{-1} + tB^{-1})^{-1}. \end{aligned}$$

In the previous paper [2], one of the authors discuss an extended path for any real number r including its extreme cases $r = \pm\infty$: For positive invertible operators A and B on a Hilbert space H, an extended path $A\, m_{r,t}\, B$ is defined as

$$A\, m_{r,t}\, B = A^{\frac{1}{2}} \left(1 - t + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}}$$

for all real numbers $r \in \mathbb{R}$ and $t \in [0,1]$. The representing function $F_{r,t}$ for $m_{r,t}$ is defined as

$$F_{r,t}(x) = 1\, m_{r,t}\, x = (1 - t + tx^r)^{\frac{1}{r}} \quad \text{for all } x > 0.$$

Notice that $A\, m_{r,t}\, B$ for $r \in \mathbb{R}$ is no longer a path of operator means for unless $-1 \le r \le 1$, but it is still interpolational and it holds the transformer equality.

In this report, we shall investigate estimates of the upper bounds for the ratio between extended interpolational paths by terms of the spectra of positive operators. Among others, if A and B are positive invertible operators on a Hilbert space H satisfying $MI \ge A, B \ge mI > 0$ for some scalars m and M and put $h = \frac{M}{m}$, then for each $t \in [0,1]$

$$r \le s \quad \text{implies} \quad A\, m_{s,t}\, B \le S(h,r,s)\, A\, m_{r,t}\, B$$

where $S(h,r,s)$ is a generalized Specht ratio.

2. A generalized Specht ratio

It is known that $F_{r,t}$ is operator monotone and operator concave for $-1 \leq r \leq 1$. But it is not for $|r| > 1$. In particular, for $r < -1$, it is concave but not operator monotone since its adjoint $(F_{r,t})^*$ coincides with $F_{-r,t}$ for $-r > 1$.

First of all, we list some properties of the representing function $F_{r,t}$ of an extended path $m_{r,t}$ for $r \in \mathbb{R}$ and $t \in [0,1]$.

Lemma 2.1. *Let $t \in (0,1)$ and $r \in \mathbb{R}$.*

(i) *$F_{r,t}(x)$ is strictly increasing and strictly convex (resp. concave) for $r > 1$ (resp. $r < 1$).*
(ii) *$F_{r,0}(x) = 1$ and $F_{r,1}(x) = x$ for all $r \in \mathbb{R}$.*
(iii) *$F_{r,t}(x) \uparrow 1 \vee x \equiv \max\{1,x\}$ as $r \uparrow \infty$.*
(iv) *$F_{r,t}(x) \downarrow 1 \wedge x \equiv \min\{1,x\}$ as $r \downarrow -\infty$.*

Proof. (i) It is increasing since

$$\frac{d}{dx}F_{r,t}(x) = tx^{r-1}\left(1-t+tx^r\right)^{\frac{1-r}{r}} > 0 \quad \text{for } t \in (0,1)$$

Moreover the latter part is shown by

$$\frac{d^2}{dx^2}F_{r,t}(x) = tx^{r-2}\left(1-t+tx^r\right)^{\frac{1-2r}{r}}(r-1)(1-t+2tx^r).$$

(ii) It suffices to show the case $r \uparrow \infty$. For the entropy function $h(x) = -x\log x$, we have

$$\frac{\partial \log F_{r,t}}{\partial r}(x) = \frac{h(1-t+tx^r) - th(x^r)}{r^2(1-t+tx^r)}.$$

The denominator is always positive and so is the numerator since Jensen's inequality shows

$$h(1-t+tx^r) > (1-t)h(1) + th(x^r) = th(x^r).$$

Thus $\log F_{r,t}(x)$ is increasing and hence so is $F_{r,t}(x)$ for $r \in \mathbb{R}$. As for convergence, we have $F_{\infty,t}(x) = 1$ for $0 < x < 1$ by

$$\lim_{r\to\infty} \log F_{r,t}(x) = \lim_{r\to\infty} \frac{\log(1-t+tx^r)}{r} = 0,$$

and $F_{\infty,t}(x) = x$ for $x \geq 1$ since l'Hospital theorem shows

$$\lim_{r\to\infty} \log F_{r,t}(x) = \lim_{r\to\infty} \frac{\log(1-t+tx^r)}{r} = \lim_{r\to\infty} \frac{tx^r \log x}{1-t+tx^r} = \log x,$$

which shows $F_{r,t}(x)$ converges to $1 \vee x$ uniformly on any finite interval.

□

Let A and B be positive invertible operators on a Hilbert space H. Since $A\ m_{r,t}\ B$ is increasing and bounded for r by Lemma 2.1, the quasi supremum $A \curlyvee B$ and quasi infimum $A \curlywedge B$ are defined as

$$A \curlyvee B = A\ m_{\infty,t}\ B \equiv \lim_{r\to\infty} A\ m_{r,t}\ B = A^{1/2}(1 \vee A^{-1/2}BA^{-1/2})A^{1/2}$$

and

$$A \mathbin{\Lsh\!\!\Rsh} B = A\ m_{-\infty,t}\ B \equiv \lim_{r\to-\infty} A\ m_{r,t}\ B = A^{1/2}(1 \wedge A^{-1/2}BA^{-1/2})A^{1/2}$$

for $t \neq 0, 1$, see [2].

Their fundamental properties as supremum and infimum in the set of all positive operators on H under the usual order are discussed in [2].

Theorem 2.2. *For each $t \in (0,1)$ an extended interpolational path $A\ m_{r,t}\ B$ is nondecreasing and norm continuous for $r \in \mathbb{R}$: For $-\infty \leq r \leq s \leq \infty$*

$$A \mathbin{\Lsh\!\!\Rsh} B = A\ m_{-\infty,t}\ B \leq \cdots \leq A\ m_{r,t}\ B \leq A\ m_{s,t}\ B \leq \cdots \leq A\ m_{\infty,t}\ B = A \mathbin{\vee\!\!\!\vee} B.$$

Proof. By (i) of Lemma 2.1, for $r \leq s$ $F_{r,t}(x) \leq F_{s,t}(x)$ implies

$$A\ m_{r,t}\ B = A^{\frac{1}{2}}F_{r,t}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \leq A^{\frac{1}{2}}F_{s,t}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A\ m_{s,t}\ B.$$

□

Thus, we shall investigate estimates of the upper bounds for the ratio between extended interpolational paths by terms of the bounds of spectra for positive operators. To prove it, we moreover need some properties of $F_{r,t}$.

For $x > 0$, put

$$f(t) = \frac{(1-t+tx^s)^{1/s}}{(1-t+tx^r)^{1/r}} \qquad (0 \leq t \leq 1)$$

and

$$S(x,r,s) = \begin{cases} \max\{f(t) : 0 \leq t \leq 1\} & \text{if } r < s, \\ \min\{f(t) : 0 \leq t \leq 1\} & \text{if } r > s. \end{cases}$$

Then we have the following result by Specht [9], and Cargo and Shisha [1]:

Lemma 2.3.

$$S(x,r,s) = \begin{cases} \left(\dfrac{r}{s-r}\dfrac{x^s-x^r}{x^r-1}\right)^{1/s}\left(\dfrac{s}{r-s}\dfrac{x^r-x^s}{x^s-1}\right)^{-1/r} & \textit{if } rs \neq 0, \\ \left(\dfrac{x^{\frac{s}{x^s-1}}}{e\log x^{\frac{s}{x^s-1}}}\right)^{1/s} & \textit{if } r = 0, \\ \left(\dfrac{x^{\frac{r}{x^r-1}}}{e\log x^{\frac{r}{x^r-1}}}\right)^{-1/r} & \textit{if } s = 0. \end{cases}$$

Proof. Let $r < s$ and $rs \neq 0$. By the elementary differential calculation, we have

$$\frac{d}{dx}\log f(t) = \frac{r((x^r-1)t+1)(x^s-1) - s((x^s-1)t+1)(x^r-1)}{sr((x^s-1)t+1)((x^r-1)t+1)},$$

and so the equation $\frac{d}{dx}\log f(t) = 0$ has the following unique solution $t = t_0$:

$$t_0 = \frac{1}{s-r}\left(\frac{r}{x^r-1} - \frac{s}{x^s-1}\right) \in (0,1).$$

Furthermore it is easily seen that

$$f'(t) > 0 \text{ for } t < t_0 \text{ and } f'(t) < 0 \text{ for } t > t_0.$$

Therefore a maximum of $f(t)$ takes at $t = t_0$, and it follows that

$$\max_{0 \le t \le 1} f(t) = f(t_0) = S(x, r, s).$$

Next, suppose the case of $r = 0 < s$. Then it follows that

$$f(t) = \frac{(1 - t + tx^s)^{1/s}}{x^t}$$

and hence we have the required result.

□

We shall investigate some properties of $S(x, r, s)$ for fixed $r \le s$ as a function $S(x) = S(x, r, s)$ for $x > 0$.

Lemma 2.4. *For given $r < s$, a function $S(x) = S(x, r, s)$ is strictly decreasing for $0 < x < 1$ and strictly increasing for $x > 1$. Furthermore the following equations hold*

$$S(1) = 1 \quad \text{and} \quad S(x) = S\left(\frac{1}{x}\right) \quad \text{for all } x > 0.$$

Proof. Since

$$\log S(x) = \frac{1}{s} \log\left(\frac{r}{s-r}\frac{x^s - x^r}{x^r - 1}\right) - \frac{1}{r}\log\left(\frac{s}{s-r}\frac{x^s - x^r}{x^s - 1}\right),$$

it follows from L'Hospital theorem that

$$\begin{aligned} \lim_{x \to 1} \log S(x) &= \lim_{x \to 1} \frac{1}{s} \log\left(\frac{r}{s-r}\frac{sx^{s-1} - rx^{r-1}}{rx^{r-1}}\right) - \frac{1}{r}\log\left(\frac{s}{s-r}\frac{sx^{s-1} - rx^{r-1}}{sx^{s-1}}\right) \\ &= \frac{1}{s}\log 1 - \frac{1}{r}\log 1 \\ &= 0 \end{aligned}$$

and so $S(1) = 1$. Also, we have $S(x) = S(\frac{1}{x})$ by direct computation.

Furthermore we have by a differential calculation

$$\frac{d}{dx}\log S(x) = x^{r-1}k(x)\frac{r(x^s - 1) - s(x^r - 1)}{sr(x^r - 1)(x^s - 1)(x^s - x^r)}$$

where

$$k(x) = (s - r)x^s - sx^{s-r} + r.$$

Suppose that $0 < r < s$. Then we have $k(x) > 0$ since $k'(x) > 0$ and $k(1) = 0$. Since $\frac{x^r - 1}{r}$ is strictly increasing for $r \in \mathbb{R}$, it follows that $0 < r < s$ implies $s(x^r - 1) < r(x^s - 1)$. Therefore we have $\frac{d}{dx}\log S(x) > 0$. Similarly we have $\frac{d}{dx}\log S(x) > 0$ in the case of $r < 0 < s$ or $r < s < 0$ and hence a function $S(x)$ is strictly increasing for all $x > 1$. On the other hand, we see that a function $S(x)$ is strictly decreasing for all $0 < x < 1$. □

Remark 2.5. *The constant $S(h)$ defined by*

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \quad (h \neq 1) \quad \text{and} \quad S(1) = 1$$

is called the Specht ratio, which is the best upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_i \in [m, M]$ with $0 < m < M$ $(i = 1, 2, \cdots, n)$,

$$\frac{x_1 + x_2 + \cdots x_n}{n} \le S(h)\sqrt[n]{x_1 x_2 \cdots x_n},$$

where the constant $h = \frac{M}{m}$ *is called a generalized condition number in the sense of Turing. By virtue of Lemma 2.4, we have for* $r \le s$ *and* $t \in [0,1]$

$$\frac{(1-t+tx^s)^{1/s}}{(1-t+tx^r)^{1/r}} \le S(h,r,s)$$

for $\frac{1}{h} \le x \le h$. *If we put* $s = 1$ *and* $r = 0$, *then the constant* $S(h,0,1)$ *coincides with the Specht ratio* $S(h)$, *i.e.,*

$$\lim_{r\to 0} S(h,r,1) = S(h).$$

Thus we call $S(x,r,s)$ *a generalized Specht ratio.*

The following lemma is an external property of a generalized Specht ratio $S(x,r,s)$:

Lemma 2.6. *Let* $r < s$.
If $x > 1$, *then* $S(x,r,s) \to x$ *as* $s \to \infty$ *or* $r \to -\infty$.
If $0 < x < 1$, *then* $S(x,r,s) \to \frac{1}{x}$ *as* $s \to \infty$.

Proof. Suppose that $x > 1$. Then

$$\left(\frac{s}{s-r}\frac{x^s - x^r}{x^s - 1}\right)^{-1/r} = \left(\frac{1}{1-\frac{r}{s}}\frac{1-x^{r-s}}{1-x^{-s}}\right)^{-1/r} \to 1$$

as $s \to \infty$ and by L'Hospital theorem

$$\begin{aligned}\lim_{s\to\infty} \log\left(\frac{r}{s-r}\frac{x^s - x^r}{x^r - 1}\right)^{1/s} &= \lim_{s\to\infty} \frac{\log\left(\frac{r}{s-r}\frac{x^s-x^r}{x^r-1}\right)}{s} \\ &= \lim_{s\to\infty} \frac{x^s}{x^s - x^r}\log x - \frac{1}{s-r} \to \log x\end{aligned}$$

Therefore we have $S(x,r,s) \to x$ as $s \to \infty$. Similarly we have $S(x,r,s) \to \frac{1}{x}$ as $s \to \infty$ for $0 < x < 1$, since $x^s \to 0$. □

On the other hand, Furuta [5] formulated the following constant as a generalized Kantorovich constant: For $h > 0$ and $p \in \mathbb{R}$

$$K(h,p) = \frac{1}{h-1}\frac{h^p - h}{p-1}\left(\frac{p-1}{h^p - h}\frac{h^p - 1}{p}\right)^p.$$

We have the following results as reverse inequalities of the Hölder-McCarthy inequality: Let A be a positive operator on H satisfying $MI \ge A \ge mI > 0$. Then

$$\begin{aligned}(A^p x, x) &\le K(h,p)(Ax,x)^p \qquad \text{for all } p \ge 1 \text{ and } p < 0,\\ K(h,p)(Ax,x)^p &\le (A^p x, x) \qquad \text{for all } 0 < p < 1\end{aligned}$$

for every unit vector $x \in H$.

We list properties of a generalized Kantorovich constant:

Lemma 2.7. *Let $h > 0$ and $p \in \mathbb{R}$.*

(1) $K(h,p) = K(\frac{1}{h}, p)$

(2) $K(h,p)$ *is increasing for $h > 1$ and decreasing for $0 < h < 1$*

(3) $K(h,p) = K(h, 1-p)$

(4) $K(1,p) = K(h,1) = K(h,0) = 1$

We have the following relation between a generalized Specht ratio and a generalized Kantorovich constant:

Lemma 2.8. *For $x > 0$ and $r, s \in \mathbb{R}$*

$$S(x,r,s) = \begin{cases} K(x^r, \frac{s}{r})^{1/s} & \text{if } rs \neq 0, \\ S(x^s)^{1/s} & \text{if } r = 0, \\ S(x^r)^{-1/r} & \text{if } s = 0. \end{cases}$$

We remark that a generalized Specht ratio $S(x,r,s)$ is the upper bound of the ratio between the power means:

Theorem 2.9. *Let A be a positive operator on H satisfying $MI \geq A \geq mI > 0$ for some scalars $M > m > 0$. Then*

$$(A^s x, x)^{1/s} \leq S(h,r,s)(A^r x, x)^{1/r} \qquad \text{for all } r < s,$$
$$S(h,r,s)(A^r x, x)^{1/r} \leq (A^s x, x)^{1/s} \qquad \text{for all } r > s$$

for every unit vector $x \in H$.

Finally, we see the Furuta formulae [5] by means of a generalized Specht ratio:

Theorem 2.10. *The following property on $K(p) = K(h,p)$ and $S(h) = S(h,0,1)$ hold.*

$$S(h) = e^{K'(1)} = e^{-K'(0)}.$$

Proof. Since $S(h,r,1) = K(h^r, \frac{1}{r}) = K(h,r)^{-1/r}$, we have

$$\begin{aligned} \log S(h) &= \lim_{r\to 0} \log S(h,r,1) = \lim_{r \to 0} -\frac{\log K(h,r)}{r} \\ &= \lim_{r\to 0} -\frac{\log K(h,r) - \log K(h,0)}{r - 0} = -\frac{K'(0)}{K(0)} = -K'(0) \end{aligned}$$

and hence $\log S(h) = -K'(0)$. On the other hand,

$$S(h,p,p+1)^{p+1} \to S(h,0,1) = S(h) \quad \text{as } p \to 0$$

and hence

$$\begin{aligned} \log S(h,p,p+1)^{p+1} &= \log K(h^p, \frac{p+1}{p}) = \log K(h, p+1)^{\frac{1}{p}} \\ &= \frac{\log K(h,p+1) - \log K(h,1)}{p+1-1} \to \frac{K'(h,1)}{K(h,1)} = K'(h,1) \end{aligned}$$

as $p \to 0$. Therefore we have $\log S(h) = K'(h,1)$. □

3. Extended interpolational path

We shall investigate estimates of the upper bounds for the ratio between extended interpolational paths $m_{r,t}$ by terms of a generalized Specht ratio. The following theorem is our main theorem.

Theorem 3.1. *Let A and B be positive operators on H such that $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r \leq s$ and $t \in (0,1)$*

$$A\ m_{s,t}\ B \leq S(h,r,s)\ A\ m_{r,t}\ B.$$

Proof. Let C be a positive invertible operator on H satisfying $MI \geq C \geq mI > 0$. Then it follow from Lemma 2.3 that

$$(1-t+tC^s)^{1/s} \leq \max_{m \leq x \leq M} S(x,r,s)(1-t+tC^r)^{1/r}$$

for all $r \leq s$ and $t \in [0,1]$. Since the maximum of $S(x,r,s)$ in $x \in [m,M]$ is given by $\max\{S(m,r,s), S(M,r,s)\}$ by Lemma 2.4, we have

$$(1-t+tC^s)^{1/s} \leq \max\{S(m,r,s), S(M,r,s)\}(1-t+tC^r)^{1/r}$$

Since $0 < mI \leq A, B \leq MI$, we obtain $\frac{1}{h}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq hI$. Replacing C by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in above inequality, we have for $t \in [0,1]$

$$(1-t+t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s)^{1/s} \leq S(h,r,s)(1-t+t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r)^{1/r}$$

since $S(\frac{1}{h},r,s) = S(h,r,s)$ by Lemma 2.4. Multiplying both sides by $A^{1/2}$, we have

$$A\ m_{s,t}\ B \leq S(h,r,s) A\ m_{r,t}\ B.$$

for $r \leq s$.

□

We investigate the order relation between the arithmetic mean, the geometric one and the harmonic one:

Corollary 3.2. *Let A and B be positive operators on H such that $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r < 0 < s$ and $t \in (0,1)$*

$$S(h^s)^{-1/s} A\ m_{s,t}\ B \leq A \sharp_t B \leq S(h^r)^{-1/r} A\ m_{r,t}\ B.$$

In particular,

$$S(h)^{-1} A \nabla_t B \leq A \sharp_t B \leq S(h) A\ !_t\ B.$$

Corollary 3.3. *Let A and B be positive operators on H such that $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r < s$ and $t \in (0,1)$*

$$\frac{1}{h} A \curlyvee B \leq A\ m_{r,s}\ B \leq A\ m_{s,t}\ B \leq h\ A \curlywedge B.$$

Lemma 3.4. *Let $x > 0$ and $r, s \in \mathbb{R}$. Then for $r < s$*

(1) $$S(x,s,r) = S(x,r,s)^{-1}.$$

(2) $$S(x,-s,-r) = S(x,r,s).$$

(3) $$S(x,r-s,r) = S(x,r,s)^{\frac{-s}{r-s}} \quad \textit{for } s > 0.$$

We investigate the order relation between extended interpolational paths.

Corollary 3.5. *Let A and B be positive operators on H such that $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r \leq s$ and $t \in (0,1)$*

(i) $S(h,s,r)\ A\ m_{s,t}\ B \leq A\ m_{r,t}\ B$.

(ii) $A\ m_{-r,t}\ B \leq S(h,r,s)\ A\ m_{-s,t}\ B$.

(iii) $A\ m_{r,t}\ B \leq S(h,r,s)^{-\frac{s}{r-s}}\ A\ m_{r-s,t}\ B$ *for* $s > 0$.

Next, we consider a generalized relative operator entropy for an extended interpolational path based on Uhlmann's method.

Lemma 3.6. *For a fixed $r \in \mathbb{R}$, $F_{r,t}$ is a convex (resp. concave) differentiable path for t if $r > 1$ (resp. $r < 1$).*

Proof.

$$\frac{\partial F_{r,t}}{\partial t}(x) = \frac{x^r - 1}{r}(1 - t + tx^r)^{\frac{1-r}{r}}$$

and

$$\frac{\partial^2 F_{r,t}}{(\partial t)^2}(x) = \frac{(x^r-1)^2(1-r)}{r^2}(1-t+tx^r)^{\frac{1-2r}{r}}.$$

□

Thereby we can define a generalized relative operator entropy including the relative operator entropy: For all real numbers $r \in \mathbb{R}$

$$S_r(A|B) = s-\lim_{t\to 0}\frac{Am_{r,t}B - A}{t} = \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r A^{\frac{1}{2}} - A}{r}$$

for positive invertible operators A and B and the representing function is $f_r(x) = (x^r - 1)/r$.

In particular, if we put $r = 0$, then $S_r(A|B)$ coincides with the relative operator entropy $S(A|B)$. We remark that a generalized relative operator entropy $S_r(A|B)$ is defined as the right differential coefficient at $t = 0$ of a function $g(t) = A\ m_{r,t}\ B$ $(r \in \mathbb{R})$.

We list several properties of a generalized relative operator entropy.

Theorem 3.7. *Let A, B, C, D be positive invertible operators on H and $r \in \mathbb{R}$. Then*

(i) *$S_r(A|B)$ is monotone increasing on $r \in \mathbb{R}$:*

$$r < s \quad \textit{implies} \quad S_r(A|B) \leq S_s(A|B).$$

(ii) *The left differential coefficient of $g(t) = A\ m_{r,t}\ B$ at $t = 1$ is $-S_r(B|A)$, i.e.,*

$$s-\lim_{t\to 1}\frac{A\ m_{r,t}\ B - B}{t-1} = -S_r(B|A).$$

(iii) *$A \leq B$ implies $S_r(A|B) \geq 0$ and $-S_r(B|A) \geq 0$.*

(iv) *If $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$, $h = \frac{M}{m}$ and $r < s$, then*

$$0 \leq S_s(A|B) - S_r(A|B) \leq \max\{\frac{h^s-1}{s} - \frac{h^r-1}{r}, \frac{h^{-s}-1}{s} - \frac{h^{-r}-1}{r}\}\ A.$$

(v) *If X is an invertible operator, then*

$$X^*S_r(A|B)X = S_r(X^*AX|X^*BX).$$

(vi) *If* $MI \geq A, B, C \geq mI > 0$ *for some scalars* $M > m > 0$, $h = \frac{M}{m}$ *and* $B \leq C$ *then for* $r > 1$

$$S_r(A|B) \leq S_r(A|C) + \frac{C(h^{-1}, h, r) - 1}{r} A$$

where

$$C(m, M, r) = \frac{mM^r - Mm^r}{M - m} \left(K(m, M, r)^{\frac{1}{r-1}} - 1\right).$$

(vii) *If* $MI \geq A, B, C, D \geq mI > 0$ *for some scalars* $M > m > 0$ *and* $h = \frac{M}{m}$, *then*

$$\begin{aligned} S_r(A + B|C + D) - \beta(A + B) &\leq S_r(A|C) + S_r(B|D) \\ &\leq S_r(A + B|C + D) + \beta(A + B) \end{aligned}$$

where

$$\beta = \max\{\frac{f_r(h) - f_r(h^{-1})}{h - h^{-1}}(x - h) + f_r(h) - f_r(x) : x \in [h^{-1}, h]\}.$$

Theorem 3.8. *A generalized relative operator entropy has an interpolational property:*

$$S_r(A|Am_{r,t}B) = tS_r(A|B)$$

for $r \in \mathbb{R}$ *and* $t \in [0, 1]$.

4. Mean-like property

Let A and B be positive invertible operators on H satisfying $MI \geq A, B \geq mI > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. An extended interpolational path $A \ m_{r,t} \ B$ ($r \in \mathbb{R}$) is no longer a path of operator mean for unless $|r| \leq 1$. In this section, we investigate several mean-like properties of an extended path. First, an extended path is subadditive in the following sense:

Theorem 4.1. *Let* $MI \geq A, B, C, D \geq mI > 0$ *and* $h = \frac{M}{m}$. *Then for* $r > 1$

$$\frac{1}{\lambda} \ (A + B) \ m_{r,t} \ (C + D) \leq A \ m_{r,t} \ C + B \ m_{r,t} \ D \leq \lambda \ (A + B) \ m_{r,t} \ (C + D)$$

where

$$\begin{aligned} \lambda &= \lambda(h^{-1}, h, F_{r,t}) \\ &= \max\{\frac{1}{F_{r,t}(x)} \left(\frac{F_{r,t}(h) - F_{r,t}(h^{-1})}{h - h^{-1}}(x - h) + F_{r,t}(h)\right) : x \in [h^{-1}, h]\}. \end{aligned}$$

Proof. Suppose that $r > 1$. Put

$$X = A^{\frac{1}{2}}(A + B)^{-\frac{1}{2}}, Y = B^{\frac{1}{2}}(A + B)^{-\frac{1}{2}}$$

and

$$V = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}, W = B^{-\frac{1}{2}}DB^{-\frac{1}{2}},$$

then we have $X^*X + Y^*Y = I$. Since $F_{r,t}$ is convex and $F_{r,t} > 0$, it follows from [8] that

$$X^*F_{r,t}(V)X + Y^*F_{r,t}(W)Y \leq \lambda F_{r,t}(X^*VX + Y^*WY).$$

Then we have

$$\begin{aligned}
&A\ m_{r,t}\ C + B\ m_{r,t}\ D \\
&= (A+B)^{\frac{1}{2}}\left(X^* F_{r,t}(V) X + Y^* F_{r,t}(W) Y\right)(A+B)^{\frac{1}{2}} \\
&\le \lambda (A+B)^{\frac{1}{2}} F_{r,t}\left(X^* V X + Y^* W Y\right)(A+B)^{\frac{1}{2}} \\
&= \lambda (A+B)^{\frac{1}{2}} F_{r,t}\left((A+B)^{-\frac{1}{2}}(C+D)(A+B)^{-\frac{1}{2}}\right)(A+B)^{\frac{1}{2}} \\
&= \lambda (A+B)\ m_{r,t}\ (C+D).
\end{aligned}$$

□

Moreover, an extended path has jointly concavity and informational monotonicity-like properties:

Theorem 4.2. *Let $MI \geq A, B, C, D \geq mI > 0$ and $h = \frac{M}{m}$. Then for $r > 1$*

$$\frac{1}{\lambda}\ (A \nabla_\alpha B)\ m_{r,t}\ (C \nabla_\alpha D) \leq (A\ m_{r,t}\ C)\ \nabla_\alpha\ (B\ m_{r,t}\ D) \leq \lambda\ (A \nabla_\alpha B)\ m_{r,t}\ (C \nabla_\alpha D)$$

where $\lambda = \lambda(h^{-1}, h, F_{r,t})$ is defined in Theorem 4.1 and $\alpha \in [0,1]$.

Proof. Suppose that $r > 1$. Since $m_{r,t}$ is homogeneous, we have by Theorem 4.1

$$\begin{aligned}
\alpha(A\ m_{r,t}\ C) + (1-\alpha)\ (B\ m_{r,t}\ D) &= (\alpha A\ m_{r,t}\ \alpha C) + ((1-\alpha)\ B\ m_{r,t}\ (1-\alpha) D) \\
&\leq \lambda(\alpha A + (1-\alpha) B)\ m_{r,t}\ (\alpha C + (1-\alpha) D).
\end{aligned}$$

□

Theorem 4.3. *Let A and B be positive invertible operators on H satisfying $MI \geq A, B \geq mI > 0$ and $h = \frac{M}{m}$. Then for $r > 1$*

$$\Phi(A\ m_{r,t}\ B) \leq \lambda\ \Phi(A)\ m_{r,t}\ \Phi(B)$$

for a normal normalized positive linear map Φ from a von Neumann algebra containing A and B to a suitable von Neumann algebra and $\lambda = \lambda(h^{-1}, h, F_{r,t})$ is defined in Theorem 4.1.

Proof. Put

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}} \Phi(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \Phi(A)^{-\frac{1}{2}},$$

then Ψ is also a normalized positive linear map. Since $F_{r,t}$ is convex for $r > 1$, it follows from converses of Jensen's inequality that

$$\lambda F_{r,t}(\Psi(X)) \geq \Psi(F_{r,t}(X)).$$

Therefore we have

$$\begin{aligned}
\Phi(A\ m_{r,t}\ B) &= \Phi(A)^{\frac{1}{2}} \Psi(F_{r,t}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})) \Phi(A)^{\frac{1}{2}} \\
&\leq \lambda \Phi(A)^{\frac{1}{2}} F_{r,t}(\Psi(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})) \Phi(A)^{\frac{1}{2}} \\
&= \lambda\ \Phi(A)\ m_{r,t}\ \Phi(B).
\end{aligned}$$

□

Finally, an extended path has an operator monotonicity-like property.

Theorem 4.4. *Let $MI \geq A, B, C, D \geq mI > 0$ such that $A \leq C$ and $B \leq D$ and $h = M/m$. Then for $r > 1$*

$$A\ m_{r,t}\ B \leq \lambda\lambda'\ C\ m_{r,t}\ D$$

where $\lambda = \lambda(h^{-1}, h, F_{r,t})$ and $\lambda' = \lambda(h^{-1}, h, F_{r,1-t})$ are defined in Theorem 4.1.

Proof.

$$\begin{aligned} A\ m_{r,t}\ B &= A^{\frac{1}{2}} F_{r,t}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \\ &\leq \lambda A^{\frac{1}{2}} F_{r,t}(A^{-\frac{1}{2}} D A^{-\frac{1}{2}}) A^{\frac{1}{2}} \\ &= \lambda\ A\ m_{r,t}\ D \\ &= \lambda\ D\ m_{r,1-t}\ A \\ &\leq \lambda\lambda'\ D\ m_{r,1-t}\ B \\ &= \lambda\lambda'\ B\ m_{r,1-t}\ D. \end{aligned}$$

□

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